Integral de linea.

16/03/12.

Tengo : 1. $f : \Omega \subset \mathbb{R}^3 \to \mathbb{R}^3$ cont en Ω

2. $\bar{x}_0$ y $\bar{x}_1$ en Ω y

$\alpha [a,b] \to \mathbb{R}^3$ curva de clase C^1 en $[a,b]$. ∋.

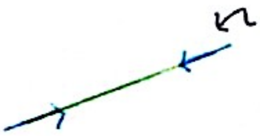
va de $\bar{x}_0$ a $\bar{x}_1$ ie $\alpha(a) = \bar{x}_0$, $\alpha(b) = \bar{x}_1$

$\wedge$ $\forall t \in [a,b]$ $\alpha(t) \in \Omega$.

Definimos $\int_\alpha f \cdot d\alpha = \int_a^b f(\alpha(t)) \cdot \alpha'(t)\, dt$.

Version sencilla del campo gravitacional.

Posición es el origen $(0,0,0)$

$\bar{x} = (x,y,z)$ 

masa m

$m > 0$ $M > 0$

Fuerza en (x,y,z) es.

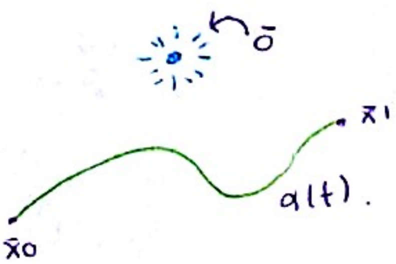
$$\bar{F}(\bar{x}) = \frac{KMm}{\|\bar{x}\|^2}\left(\frac{-\bar{x}}{\|\bar{x}\|}\right) = -KMm\left(\frac{x}{\|\bar{x}\|^3}, \frac{y}{\|\bar{x}\|^3}, \frac{z}{\|\bar{x}\|^3}\right)$$

K cte y >0

Además $F : \mathbb{R}^3 \setminus \{\bar{0}\} \to \mathbb{R}^3$

Sean $\bar{x}_0, \bar{x}_1 \in \mathbb{R}^3 \setminus \{\bar{0}\}$.

y α una curva que va de $\bar{x}_0$ a $\bar{x}_1$ contenido en $\mathbb{R}^3 \setminus \{\bar{0}\}$.



Quiero $\int_\alpha F\, d\alpha$.

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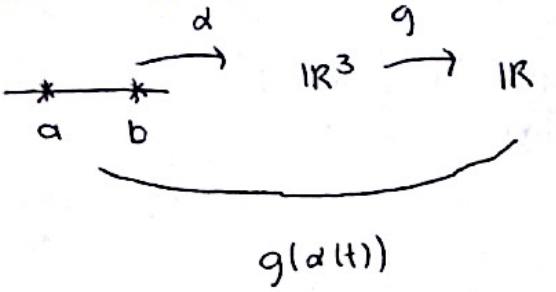
$$T(x,y,z) = \left(\frac{-KMmx}{(x^2+y^2+z^2)^{3/2}}, \frac{-KMmy}{(x^2+y^2+z^2)^{3/2}}, \frac{-KMmz}{(x^2+y^2+z^2)^{3/2}} \right)$$
$$= (F_1(\bar{x}), F_2(\bar{x}), F_3(\bar{x}))$$

Calculo III



$g \circ \alpha : [a,b] \to \mathbb{R}.$

$$\frac{d}{dt} (g \circ \alpha)(t) = \nabla g(\alpha(t)) \cdot \alpha'(t)$$

Si $f = \nabla g \rightarrow \int_a^b \frac{d}{dt}(g \circ \alpha)\, dt.$

$\cdot\, \alpha'(t) \qquad g \circ \alpha(t) \Big|_a^b$

¿Es $F: \mathbb{R}^3 \setminus \{\bar{0}\} \to \mathbb{R}^3$ el gradiente de $G: \mathbb{R}^3 \setminus \{\bar{0}\} \to \mathbb{R}^3$.

Si si

$$\frac{G}{\partial x}\Big|_{\bar{x}} = F_1(\bar{x}), \quad \frac{\partial G}{\partial y}\Big|_{\bar{x}} = F_2(\bar{x}), \quad \frac{\partial G}{\partial z}\Big|_{\bar{x}} = F_3(\bar{x})$$

$$\frac{-KMmx}{(x^2+y^2+z^2)^{3/2}} \rightarrow G(x,y,z) = \int \frac{-KMmx}{(x^2+y^2+z^2)^{3/2}}\, dx.$$
$$= KMm\,(x^2+y^2+z^2)^{-1/2}.$$

$$F_1(x,y,z) = \frac{-KMmx}{(x^2+y^2+z^2)^{3/2}} \xrightarrow[c/x]{\text{integrar}} G(x,y,z) = \frac{KMm}{(x^2+y^2+z^2)^{1/2}} + \varphi_1(y,z).$$

$$F_2(x,y,z) = \frac{-KMmy}{(x^2+y^2+z^2)^{3/2}} \xrightarrow{c/y} G(x,y,z) = \frac{KMm}{(x^2+y^2+z^2)^{1/2}} + \varphi_2(x,z)$$

$$F_3(x,y,z) = \frac{-KMmz}{(x^2+y^2+z^2)^{3/2}} \xrightarrow{c/z} G(x,y,z) = \frac{KMm}{(x^2+y^2+z^2)^{1/2}} + \varphi_3(x,y).$$

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Consideras

$$G(x,y,z) = \frac{KMm}{(x^2+y^2+z^2)^{1/2}}$$

$$\nabla G(\bar{x}) = F(\bar{x})$$

$$G : \mathbb{R}^3 \setminus \{\bar{0}\} \to \mathbb{R}.$$

Regresando. ent.

$$\int_\alpha F\, d\alpha = \int_a^b F(\alpha(t)) \cdot \alpha'(t)\, dt$$

$$= \int_a^b \nabla G(\alpha(t)) \cdot \alpha'(t)\, dt.$$

$$= \int_a^b \frac{d}{dt} (G(\alpha(t))\, dt.$$

$$= G(\alpha(t)) \Big|_a^b = G(\alpha(b)) - G(\alpha(a))$$

Por lo tanto

$$\int_\alpha F\, d\alpha = G(\bar{x}_1) - G(\bar{x}_0)$$
$$= KMm \left(\frac{1}{\|\bar{x}_1\|} - \frac{1}{\|\bar{x}_0\|} \right)$$

Obs.

1. $\int_\alpha F\, d\alpha$ no depende de α, sólo de $\bar{x}_0 \wedge \bar{x}_1$

2. i) $\|\bar{x}_0\| = \|\bar{x}_1\| \Rightarrow \int_\alpha F\, d\alpha = 0$

3. $\int_\alpha f\, d\alpha = KMm \left(\frac{1}{\|\bar{x}_1\|} - \frac{1}{\|\bar{x}_0\|} \right)$

? Si $x_1 \to \bar{0}$
$\int_\alpha f\, d\alpha \longrightarrow -\infty$

? Si $\bar{x}_1 \to \infty$
$\int_\alpha f\, d\alpha \longrightarrow \frac{-KMm}{\|\bar{x}_0\|}$

ocio

16/03/12

Propiedades de la integral de línea.

$f: \Omega \subset \mathbb{R}^3 \to \mathbb{R}^3$

Ω abierto y conexo por trayectorias f cont en Ω

Definición Decimos que f es un campo gradiente si $\exists\ g: \Omega \subset \mathbb{R}^3 \to \mathbb{R}$

$g \in C^1$ en Ω $\cdot\ni\cdot$

$\forall \bar{x} \in \Omega \quad \nabla g(x) = f(x)$

ie $\left(\frac{\partial g}{\partial x}\Big|_{\bar{x}}, \frac{\partial g}{\partial y}\Big|_{\bar{x}}, \frac{\partial g}{\partial z}\Big|_{\bar{x}} \right) = (f_1(\bar{x}), f_2(\bar{x}), f_3(\bar{x}))$

Prop. Si f es un campo gradiente en Ω ie $\nabla g = f$ en todo Ω ent $\forall\ \bar{x}_0, \bar{x}_1$ en Ω y α una curva que va de $\bar{x}_0$ a $\bar{x}_1$ se tiene. que

$$\int_\alpha f \cdot d\alpha = g(\bar{x}_1) - g(\bar{x}_0)$$

Dem

$$\int_\alpha f \cdot d\alpha = \int_a^b f(\alpha(t)) \cdot \alpha'(t)\, dt$$

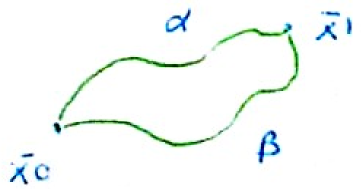
$$= \int_a^b \nabla g(\alpha(t)) \cdot \alpha'(t)\, dt.$$

$$= \int_a^b \frac{d}{dt} (g \circ \alpha(t))\, dt.$$

$$= g \circ \alpha(t) \Big|_a^b = g(\alpha(b)) - g(\alpha(a))$$

$$= g(\bar{x}_1) - g(\bar{x}_0)$$

Corolario Si f es campo gradiente y



ent

$$\int_\alpha f \cdot d\alpha = \int_\beta f \cdot d\beta$$

"Ah, ent la integral de linea no depende de la trayectoria"

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Corolario Si $\exists\ \bar{x}_0, \bar{x}_1 \in \Omega$ y α, β de $\bar{x}_0$ a $\bar{x}_1$ $\ni$

$$\int_\alpha f \cdot d\alpha \neq \int_\beta f \cdot d\beta \Rightarrow f \text{ no es gradiente.}$$

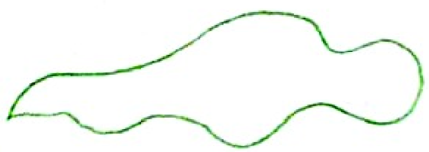
Campos gradientes	Campos no gradientes
$f(x,y,z) = (0,0,K) = \nabla g(x,y,z)$ $g(x,y,z) = Kz.$	$f(x,y,z) = (y,0,0)$ $\exists\ \gamma, \beta$ del $(1,0,0)$ al $(-1,0,0)$ $\ni$ $\int_\alpha f \cdot d\alpha \neq \int_\beta f \cdot d\beta$
$f(x,y,z) = (K_1, K_2, K_3) = \nabla g(x,y,z)$ $g(x,y,z) = K_1x + K_2y + K_3z$	

$$f(x,y,z) = \frac{-x}{(x^2+y^2+z^2)^{3/2}}, \frac{-y}{(x^2+y^2+z^2)^{3/2}}, \frac{-z}{(x^2+y^2+z^2)^{3/2}}$$

$$g(x,y,z) = \frac{1}{(x^2+y^2+z^2)^{1/2}}$$

Def Sea $\alpha: [a,b] \to \mathbb{R}^3$ una curva

Decimos que α es una curva cerrada simple $\alpha(a) = \alpha(b)$



$\alpha(t)$

$\alpha(a) = \alpha(b)$

Si $\alpha(s) = \alpha(t)$, $s \neq t) \Rightarrow s,t \in \{a,b\}$

Proposic Si f es un campo gradiente en $\forall$ curva cerrada simple $\{\alpha(t)\} \subset \Omega$ se tiene que.

$$\int_\alpha f \cdot d\alpha = 0$$

Dem

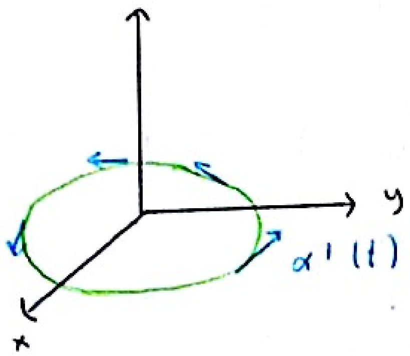
$$\int_\alpha f \cdot d\alpha = g(\alpha(b)) - g(\alpha(a)) = 0$$

ya que $\alpha(a) = \alpha(b)$

Corolario Si $\exists$ una curva cerrada simple α en Ω $\ni$ ·

$$\int_\alpha f\, d\alpha \neq 0 \Rightarrow f \text{ no es un campo gradiente.}$$

Campo famoso no gradiente.



$$f(x,y,z) = (-y, x, 0)$$
$$\alpha(t) = (\cos t, \operatorname{sen} t, 0) \quad t \in [0, 2\pi]$$

$$\int_\alpha f \cdot d\alpha = \int_0^{2\pi} (-\operatorname{sen} t, \cos t, 0) \cdot (-\operatorname{sen} t, \cos t, 0)\, dt.$$

$$= \int_0^{2\pi} \operatorname{sen}^2 t + \cos^2 t)\, dt.$$

$$= \int_0^{2\pi} 1\, dt = 2\pi \neq 0.$$

$$\beta(t) = (0, \cos t, \operatorname{sen} t) \quad t \in [0, 2\pi]$$

$$\int_\beta f \cdot d\beta = \int_0 (-\cos t, 0, 0) \cdot (0, -\operatorname{sen} t, \cos t)\, dt = 0.$$